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THE NIL RADICAL OF POWER SERIES RINGS

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ABSTRACT

We describe the nil radical of power series rings in non-commuting indeterminates by showing that a series belongs to the radical if and only if the ideal generated by its coefficients is nilpotent. We also show that the principal ideals generated by elements of the nil radical of the power series ring in one indeterminate are nil of bounded index.

Introduction

Many studies in the theory of associative rings concern radicals and related properties of rings arising under various constructions. They are rich in interesting and useful results but there are still many open problems in the area (cf. [1, 10] and the papers cited therein). These in particular concern power series rings (see, for instance, [1, 6, 7, 8]). In this paper we continue studies of the nil radical of such rings.

It is relatively easy to show [8] that the power series ring in at least two non-commuting indeterminates is nil if and only if the coefficient ring is nilpotent. Thus it is natural to expect ([10], Question 16) that an element of a power series ring in at least two non-commuting indeterminates belongs to the nil radical of

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the ring if and only if the ideal of the coefficient ring generated by the coefficients of the element is nilpotent. We shall prove that this is indeed the case.

The structure of the nil radical of power series rings in one indeterminate is more complicated. In [4] Klein proved that if R is a nil ring of bounded index, then so is the ring $R[x]$ of polynomials in one indeterminate x. This immediately implies that also the power series ring $R\{x\}$ is nil of bounded index. In [8] it was proved that if $R\{x\}$ is a nil ring, then R is nil of bounded index. Thus $R\{x\}$ is nil if and only if R is nil of bounded index and one could expect ([10], Question 15) that an element of the power series ring $A\{x\}$ belongs to the nil radical of $A\{x\}$ if and only if the ideal of A generated by the coefficients of the element is nil of bounded index. An example in [2] shows that this is not true even for commutative rings. However, we shall prove that every principal right nil ideal of $A\{x\}$ is nil of bounded index. This in particular shows that the nil radical of $A\{x\}$ coincides with $N(A\{x\})$, where for a ring R, $N(R)$ is defined ([11], p. 206) as $N(R) = \{r \in R \mid rR \text{ is nil of bounded index }\}$.

1. Some sequences of natural numbers

Throughout the paper $\mathbb{N}$ denotes the set of natural numbers and $\mathbf{S}$ denotes the set of all sequences $(n_i)_{i=1}^{\infty}$ of natural numbers such that $n_1 = 1$ and $n_{i+1} = 3^{p_i}$ for $i \geq 1$, where p_i is a natural number with $n_i + 2i \leq p_i \leq n_i + 3i$.

LEMMA 1: *Let $(n_i), (m_i) \in \mathbf{S}$. If for some k, l, $n_k = m_l$, then $k = l$ and $n_i = m_i$ for $i \leq k$.*

Proof: Let $u_1 = 3$, $v_1 = 4$ and, for $i \geq 1$, $u_{i+1} = 3^{u_i} + 2(i+1)$, $v_{i+1} = 3^{v_i} + 3(i+1)$. Clearly for all i, $u_i < v_i$. We shall prove that for each i, $v_i + i < u_{i+1}$. It is clear for $i = 1$. Note that for each $i \geq 1$, $3^{v_i} > 1$ and $3^i \geq 2i+1$. Hence if $v_i + i < u_{i+1}$, then $v_{i+1} + i + 1 = 3^{v_i} + 4(i+1) = 3^{v_i} + 2i + 2(i+2) \leq 3^{v_i} + 3^i - 1 + 2(i+2) < 3^{v_i} + 3^{v_i}(3^i - 1) + 2(i+2) = 3^{v_i+i} + 2(i+2) < 3^{u_{i+1}} + 2(i+2) = u_{i+2}$. Hence the inequality follows by induction. In particular it shows that the intervals $[u_i, v_i]$ are disjoint.

We shall prove that if $n_{i+1} = 3^{p_i}$, then $u_i \leq p_i \leq v_i$. It is clear for $i = 1$. Now $3^{p_i} + 2(i+1) = n_{i+1} + 2(i+1) \leq p_{i+1} \leq n_{i+1} + 3(i+1) = 3^{p_i} + 3(i+1)$. Hence if $u_i \leq p_i \leq v_i$, then $u_{i+1} = 3^{u_i} + 2(i+1) \leq 3^{p_i} + 2(i+1) \leq p_{i+1} \leq 3^{p_i} + 3(i+1) \leq 3^{v_i} + 3(i+1) = v_{i+1}$. Hence the inequalities follow by induction.

The foregoing prove the first claim of the lemma. To get the second claim it suffices to show that for each i, $n_i > m_i$ implies $n_{i+1} > m_{i+1}$. Note that since n_i and m_i are powers of 3, if $n_i > m_i$, then $n_i \geq 3m_i$. Now if $n_{i+1} = 3^{p_i}$ and

$m_{i+1} = 3^{q_i}$, then $p_i \geq n_i + 2i \geq 3m_i + 2i = m_i + 2m_i + 2i > m_i + i + 2i = m_i + 3i \geq q_i$. Consequently $n_{i+1} > m_{i+1}$ and the result follows. ∎

We shall say that a natural number t **belongs to** a sequence $(s_i) \in \mathbf{S}$ if $t = s_i$ for some i. Given natural numbers t and u , we call u an **S-successor** of t if there is $(s_i) \in \mathbf{S}$ such that $s_k = t$ and $s_l = u$ for some $k \leq l$.

LEMMA 2: *If $\mathbf{S} = \bigcup_{l=1}^{\infty} \mathbf{S}_l$, for some subsets $\mathbf{S}_l$, then there exist $n \in \mathbb{N}$ and t belonging to a sequence of $\mathbf{S}$ such that each $\mathbf{S}$-successor of t belongs to a sequence of $\mathbf{S}_n$.*

Proof: Suppose the result does not hold, i.e., for each t belonging to a sequence of $\mathbf{S}$ and each i there exists an $\mathbf{S}$-successor $f_i(t)$ of t such that $f_i(t)$ does not belong to any sequence of $\mathbf{S}_i$.

Put $t_1 = 1$ and, for $i \geq 1$, $t_{i+1} = f_i(t_i)$. Applying Lemma 1 one gets that $t_1, t_2, \ldots$ is a subsequence of a sequence $s \in \mathbf{S}$. It is clear that $s \notin \bigcup_{l=1}^{\infty} \mathbf{S}_l$, a contradiction. ∎

2. Some properties of free monoids

Let W be the free monoid generated by a set of non-commuting indeterminates containing x and y. As usual, given $w \in W$, we denote by $l(w)$ the length of w.

Given a subset X of the set $\{xy^kx | \ k \in \mathbf{N}\}$, define for $w \in W$

$$d_X(w) = \operatorname{card}\{(w_1, w_2) \in W \times W \mid xwx = w_1 u w_2 \text{ for some } u \in X\}.$$

Clearly $d_{\emptyset}(w) = 0$ for every $w \in W$.

LEMMA 3: *The mapping $d_X(w)$ has the following properties.*

(a) *If $w = xy^kx$, $k \in \mathbf{N}$, then $d_X(y^k) = d_X(w) = \begin{cases} 0 & \text{if } w \notin X, \\ 1 & \text{if } w \in X. \end{cases}$*

(b) *If $q_1, q_2 \in W$ and $u \in \{xy^kx | \ k \in \mathbf{N}\}$, then $d_X(q_1uq_2) = d_X(q_1) + d_X(u) + d_X(q_2)$.*

(c) *If $p_1up_2 = q_1u'q_2$ for some $u, u' \in X$ and $p_1, p_2, q_1, q_2 \in W$, then $d_X(p_1) = d_X(q_1)$ implies that $p_1 = q_1$, $u = u'$, $p_2 = q_2$.*

Proof: The property (a) is a direct consequence of the definition of d_X. To get (b) let us observe that given $p_1, p_2 \in W$, $xp_1xp_2x = w_1xy^kxw_2$ for some $w_1, w_2 \in W$ if and only if precisely one of the equalities $xp_1x = w_1xy^kxw_1'$ or $xp_2x = w_2'xy^kxw_2$ is satisfied for some $w_1', w_2' \in W$. Hence $d_X(p_1xp_2) = d_X(p_1) + d_X(p_2)$. Consequently, if $u = xy^kx$, then $d_X(q_1uq_2) = d_X(q_1) + d_X(y^k) + d_X(q_2) = d_X(q_1) + d_X(u) + d_X(q_2)$. Now we shall prove (c). Note first that by

(b) we have $d_X(p_2) = d_X(q_2)$. Observe also that if $l(p_1) < l(q_1)$, then $p_1uq' = q_1$ for some $q' \in W$ or $p_1u = q_1x$. In both cases, applying (a) and (b), we get that $d_X(p_2) \neq d_X(q_2)$, a contradiction. Symmetric arguments show that also $l(p_1) > l(q_1)$ is impossible. Thus $l(p_1) = l(q_1)$ and one easily concludes the proof. ∎

LEMMA 4: *Let X and Y be subsets of the set $\{xy^kx|\ k \in \mathbb{N}\}$. Consider elements $p_1,...,p_n$, $q_1,...,q_n$ of W such that $d_{X\setminus Y}(p_i) = 0$ for $i = 1,...,n$. If $p_1x_1p_2x_2\cdots p_nx_n = q_1x'_1q_2x'_2\cdots q_nx'_n$, where $x_1,...,x_n \in X\setminus Y$ and $x'_1,\dots,x'_n \in X$, then $p_1 = q_1,...,p_n = q_n$, $x_1 = x'_1,...,x_n = x'_n$ or $\sum_{i=1}^n d_Y(q_i) < \sum_{i=1}^n d_Y(p_i)$.*

Proof: By Lemma 3 (a),(b) we have

$$\begin{aligned} d_Y(p_1) + d_Y(p_2) + \cdots + d_Y(p_n) &= d_Y(p_1x_1\cdots p_nx_n) = d_Y(q_1x'_1\cdots q_nx'_n) \\ &= d_Y(q_1) + \cdots + d_Y(q_n) + d_Y(x'_1) + \cdots + d_Y(x'_n). \end{aligned}$$

Thus $\sum_{i=1}^n d_Y(q_i) \leq \sum_{i=1}^n d_Y(p_i)$. Moreover, if $\sum_{i=1}^n d_Y(q_i) = \sum_{i=1}^n d_Y(p_i)$, then $d_Y(x'_1) = \cdots = d_Y(x'_n) = 0$, so $x'_1,\dots,x'_n \in X\setminus Y$. Now applying again Lemma 3 (a),(b) we get that

$$\begin{aligned} n = d_{X\setminus Y}(x_1) + \cdots + d_{X\setminus Y}(x_n) &= d_{X\setminus Y}(p_1x_1\cdots p_nx_n) \\ &= d_{X\setminus Y}(q_1x'_1\cdots q_nx'_n) \\ &= d_{X\setminus Y}(q_1) + \cdots + d_{X\setminus Y}(q_n) + n. \end{aligned}$$

Consequently, $d_{X\setminus Y}(q_1) = \cdots = d_{X\setminus Y}(q_n) = 0$. Applying Lemma 3 (c) to $d_{X\setminus Y}$ we get by induction that $p_1 = q_1,...,p_n = q_n$ and $x_1 = x'_1,...,x_n = x'_n$. ∎

3. Nil ideals of power series rings

We start with

LEMMA 5: *Let A be a subset of a ring R. The right ideal of R generated by A is nilpotent if and only if for every finite or countable subset B of R there is an n such that $a_1b_1\cdots a_nb_n = 0$ for arbitrary $a_1,a_2,...,a_n \in A$ and $b_1,...,b_n \in B$.*

Proof: The "only if" part of the result is clear. If the "if" part does not hold, then for every n there exist $a_{1,n},\dots,a_{n,n} \in A$ and $r_{1,n},\dots,r_{n,n} \in R$ such that $a_{1,n}r_{1,n}\cdots a_{n,n}r_{n,n} \neq 0$. However, then for $B = \{r_{i,j}\}$ the assumption does not hold. ∎

Let, like in Section 2, W denote the free monoid generated by a set of indeterminates containing x and y. The power series ring over a ring R in this set of indeterminates will be denoted by $R\{W\}$. Every element $a \in R\{W\}$ can be uniquely presented in the form $\sum_{p\in W} a_p p$, where $a_p \in R$. Elements a_p are called coefficients of a.

THEOREM 6: *An element $a = \sum_{p\in W} a_p p$ belongs to a right nil ideal of $R\{W\}$ if and only if the right ideal of R generated by the set $\{a_p \mid p \in W\}$ of coefficients of a is nilpotent.*

Proof: Let $b_0, b_1, \ldots$ be any elements of R. Put, for every odd integer $q > 0$, $r_{q2^k} = b_k$, $k = 0, 1, \ldots$, and $\mathbf{S}_l = \{(s_i) \in \mathbf{S} | \ (a\sum_{i=1}^{\infty} r_i x y^{s_i} x)^l = 0\}$. Since a is in a right nil ideal of $R\{W\}$, we have that $\mathbf{S} = \bigcup \mathbf{S}_l$. Let t and n be given by Lemma 2 applied to these sets. Suppose that $t = c_f$ belongs to a sequence $(c_i) \in \mathbf{S}_n$ and put $C = \{xy^{c_1}x, xy^{c_2}x, \ldots\}$.

By Lemma 5 it suffices to show that for arbitrary coefficients $a_{p_1}, \ldots, a_{p_n}$ of a and arbitrary non-negative integers $k_1, \ldots, k_n$, we have $a_{p_1} b_{k_1} \cdots a_{p_n} b_{k_n} = 0$. We proceed by induction on $j = d_C(p_1) + \cdots + d_C(p_n)$. If $j = 0$, then $d_C(p_1) = \cdots = d_C(p_n) = 0$. Hence applying Lemma 4 for $X = C$ and $Y = \emptyset$, we get that for arbitrary non-negative integers $i_1, \ldots, i_n$ the coefficient at $p_1 x y^{c_{i_1}} x \cdots p_n x y^{c_{i_n}} x$ in the series $(a \sum r_i x y^{c_i} x)^n$ is equal to $a_{p_1} r_{i_1} \cdots a_{p_n} r_{i_n}$. But, on the other hand, the coefficient is equal to zero. Hence taking $i_1 = 2^{k_1}, \cdots, i_n = 2^{k_n}$ we get that $a_{p_1} b_{k_1} \cdots a_{p_n} b_{k_n} = 0$. Suppose now that $j > 0$ and the result holds for smaller integers. Let $m = \max(f, l(p_1 \cdots p_n))$. By the definition of sequences in $\mathbf{S}$ there exists a sequence $(d_i') \in \mathbf{S}$ such that $d_i' = c_i$ for $i \leq m$ and $d_{m+1}' \neq c_{m+1}$. Clearly d_{m+1}' is an $\mathbf{S}$-succesor of t, so by Lemma 2 there exists $(d_i) \in \mathbf{S}_n$ such that for some k, $d_k = d_{m+1}'$. Applying Lemma 1 to (d_i) and (d_i') we get that $k = m + 1$ and $d_i' = d_i$ for $i \leq m + 1$, and applying the lemma to (d_i) and (c_i) we obtain that $d_i = c_i$ if and only if $i \leq m$. Note that since $l(p_1 \cdots p_n) \leq m$, for $Y = C$ and $X = \{xy^{d_1}x, xy^{d_2}x, \ldots\}$, $d_{X \setminus Y}(p_i) = 0$ for $i = 1, \ldots, n$. Hence applying Lemma 4 and the induction assumption we get that for any $i_1, \ldots, i_n > m$ the coefficient at $p_1 x y^{d_{i_1}} x \cdots p_n x y^{d_{i_n}} x$ in the series $(a \sum r_i x y^{d_i} x)^n$ is equal to $a_{p_1} r_{i_1} \cdots a_{p_n} r_{i_n}$. On the other hand, the coefficient is equal to zero. Hence taking $i_1 = (2m+1)2^{k_1}, \ldots, i_n = (2m+1)2^{k_n}$ we get that $a_{p_1} b_{k_1} \cdots a_{p_n} b_{k_n} = 0$. ∎

Given a ring A, denote by $N(A)$ the Wedderburn radical of A, i.e., the sum of all nilpotent ideals of A, and by $N_1(A)$ the ideal of A containing $N(A)$ such that $N_1(A)/N(A) = N(A/N(A))$.

Theorem 6 gives immediately

COROLLARY 7: *The nil radical of $R\{W\}$ is equal to $N(R\{W\})$.*

Now we pass to power series rings $R\{x\}$ in one indeterminate x with coefficients in a ring R. The ring of polynomials in the indeterminate x over R will be denoted by $R[x]$. As usual, the degree of a non-zero polynomial $f(x)$ is denoted by $\deg f(x)$. Moreover we put $\deg 0 = 0$.

LEMMA 8: *Let $a(x)$ be an element of a right nil ideal of $R\{x\}$. There exist a natural number n and $f(x) \in R[x]$ such that for every $p(x) \in R[x]$ there is $b(x) \in R\{x\}$ with $(a(x)(f(x) + p(x)x^{\deg f(x)} + b(x)x^{\deg f(x) + \deg p(x)}))^n = 0$.*

Proof: Suppose the result does not hold. Then for every natural number n there is a polynomial $g_n(x)$ such that for every $b(x) \in R\{x\}$,

$$(a(x)(f_n(x) + g_n(x)x^{\deg f_n(x)} + b(x)x^{\deg f_n(x) + \deg g_n(x)}))^n \neq 0,$$

where $f_1(x) = 0$ and for $n \geq 1$, $f_{n+1}(x) = f_n(x) + g_n(x)x^{\deg f_n(x)}$. Now it is not hard to check that for

$$a^*(x) = f_1(x) + g_1(x)x^{\deg f_1(x)} + g_2(x)x^{\deg f_2(x)} + \cdots \in R\{x\}$$

we have $(a(x)a^*(x))^n \neq 0$ for every natural number n, a contradiction. ∎

THEOREM 9: *If $a(x)$ belongs to a right nil ideal of $R\{x\}$, then there is a natural number n such that $a(x)R\{x\}$ is nil of index $\leq n$.*

Proof: Let n and $f(x)$ be those of Lemma 8. It is not hard to see that it suffices to prove that for every $g(x) \in R[x]$, we have $(a(x)g(x))^n = 0$. Take any $g_0(x) \in R[x]$. Applying Lemma 8 for $p(x) = g_0(x) + rx^m$, where r is a non-zero element of R and m is a natural number, we get that $(a(x)(f(x) + g_0(x)x^{\deg f(x)}))^n \in x^m R\{x\}$. Since we can take m arbitrary and $(a(x)(f(x) + g_0(x)x^{\deg f(x)})^n$ does not depend on m, we get that $(a(x)(f(x) + g_0(x)x^{\deg f(x)}))^n = 0$.

Substituting in the foregoing $g_0(x) = g(x)x^{l - \deg f(x)}$ with $l = \deg f(x) + k$, we get that $(a(x)(f(x) + g(x)x^k))^n = 0$ for $k = 1, 2, \ldots$. Hence

$$(a(x)f(x))^n + a_1(x)x^k + a_2(x)x^{2k} + \cdots + a_{n-1}(x)x^{(n-1)k} + (a(x)g(x))^n x^{nk} = 0,$$

for some $a_1(x), \ldots, a_{n-1}(x) \in R\{x\}$ not depending on k. This gives $(a(x)f(x))^n = a_1(x) = \cdots = a_{n-1}(x) = (a(x)g(x))^n = 0$. The result follows. ∎

In [5], Theorem 2, it was proved that for every ring A the set $\{a \in A \mid$ the right ideal of A generated by a is nil of bounded index$\}$ is contained in $N_1(A)$. This and Theorem 9 give

COROLLARY 10: *The nil radical of $R\{x\}$ is equal to $N_1(R\{x\})$.*

The following theorem was proved in [2]. In particular it shows that there exists a commutative algebra A and an element in the nil radical of $A\{x\}$ such that the subalgebra of A generated by the coefficients of the element is not nil of bounded index. We shall give a slightly simpler proof of the theorem.

THEOREM 11 ([2], Theorem 3.1): *Let F be the free commutative K-algebra generated by indeterminates $s_1, s_2, \ldots$ over a field K of characteristic zero. Let I be the ideal of F generated by elements*

$$h_k = \sum_{i=1}^{k} s_i s_{k-i}, \quad k = 2, \ldots$$

and $A = F/I$. Then A is not nil of bounded index and $(\bar{s}_1 x + \bar{s}_2 x^2 + \cdots)^2 = 0$ in $A\{x\}$, where $\bar{s}_1 = s_1 + I, \bar{s}_2 = s_2 + I, \ldots$.

Proof: Clearly h_k are the coefficients of the series $(s_1 x + s_2 x^2 + \cdots)^2$, so indeed $(\bar{s}_1 x + \bar{s}_2 x^2 + \cdots)^2 = 0$.

Given a natural number $m \geq 3$, let J_m be the ideal of F generated by all monomials $s_{i_1} \cdots s_{i_t}$ such that $t > m$ or $i_1 + \cdots + i_t > m^{m^2}$. Note that $\{s_{i_1} \cdots s_{i_m} + J_m \mid 1 \leq i_1 \leq m^{m^2-2}, m^{m^2-2} < i_2 \leq 2m^{m^2-2}, 2m^{m^2-2} < i_3 \leq 3m^{m^2-2}, \ldots, (m-1)m^{m^2-2} < i_m \leq m^{m^2-1}\}$ is a K-linearly independent subset of $(F^m + J_m)/J_m$. Consequently $\dim_K((F^m + J_m)/J_m) \geq (m^{m^2-2})^m = m^{m(m^2-2)}$. On the other hand, $(I + J_m)/J_m$ is generated as a linear space over K by the set $\{h_k + J_m \mid k \leq m^{m^2}\} \cup \{h_k s_{i_1} \cdots s_{i_t} + J_m \mid k, i_1, \ldots, i_t \leq m^{m^2}, t \leq m-2\}$. Hence $\dim_K((I + J_m)/J_m) \leq m^{m^2} + (m^{m^2})^2 + \cdots + (m^{m^2})^{m-1} < m(m^{m^2})^{m-1} = m^{(m-1)m^2+1}$. However, for $m \geq 3$, we have $m(m^2-2) > (m-1)m^2 + 1$, so $F^m + J_m/J_m \not\subseteq I + J_m/J_m$. This shows that $A = F/I$ is not nilpotent and hence, by the Nagata–Higman theorem [3], A is not nil of bounded index. ∎

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